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Self-Dual Transitive Spin 1/2 Models

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Abstract Most general self-dual spin 1/2 models in any dimension, with interaction that is translation-invariant in a suitable sense ('transitive models'), are determined. In the process of classification of such systems, a class of models which are self-dual in a particularly strong sense is introduced.

Keywords Self-duality · Spin 1/2 models

1 Introduction

Starting with Kramers-Wannier [11] and Onsager [13, 21], duality arguments have been used extensively in Statistical Mechanics of a lattice models. Restricted at first to systems with two-body nearest neighbor interactions (Ising Model on various lattices, Potts Model), in 1970s the theory has been extended to systems in which any number of spins may be coupled (Wegner [7, 12, 22, 24]).

Duality yields especially strong results in case of self-dual systems. And, starting again with [11], quite a few models has been shown to be self-dual—to some of these we will refer below.

The term 'duality' is used in Statistical Mechanics to describe a relation between a system at low temperature and another one at high temperature, at large and small values of coupling constants in Field Theory. Such a relation can be obtained by many different methods: transfer matrix method of [11]; 'topological' method of Onsager, and the equivalent graph theoretical method; combination of these methods with other transformations, star-triangle, for example, again originating with Onsager; and a method introduced by Wegner [22] that uses vector space structure over the two-element field $\mathbb{F}_2 = \{0, 1\}$ (or the corresponding group structure [7, 12]). We stress that the present paper is concerned with spin 1/2 systems only. Duality in systems with more than two components, like the Potts model, or lattice

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field theory—see [14, 25] for early reviews, and [5] for a more recent publication—is not discussed here. Also, we concentrate on the description of self-dual models, leaving out the standard by now consequences of duality, relation between order and disorder variables, in particular, for which we refer to [7, 9, 10, 22].

The notion of duality adopted here is that of [22] (and of [7]). It is equivalent, in a sense, to that of theory of binary linear codes, [19]. It is general enough to cover a host of interesting examples (in any dimension; systems with finite and infinite number of ground states and low-temperature phases), and restrictive enough to allow for a general analysis: we describe, for any dimension d, the most general self-dual spin 1/2 model on the simple lattice $\mathbb{Z}^d$, with interaction that is $\mathbb{Z}^d$-invariant in a suitable sense ('transitive models', for short: the two-dimensional Ising Model and the triangular three-spin model—models M_1 and M_3 of Sect. 2, are transitive, while the (self-dual Type A) Union-Jack model, or the $M_{2n,n}$ models, $n \geq 2$, of [22], and the model of [23], are not, and are outside the scope of the present paper). Extensions of our results to general lattices and to Abelian groups different from $\mathbb{F}_2$—to situations where the constructions are much less canonical, will be treated in a future publication.

Our analysis is facilitated by considering algebraic structure which incorporates the translations, i.e., the structure of a $\mathbb{F}_2[\mathbb{Z}^d]$-module introduced into the present context in [8] (Sect. 3 below), with the result that the infinite dimensional vector space (over the field $\mathbb{F}_2$) of infinite systems is turned into a finite dimensional module, or even into a finite dimensional vector space over the field of fractions of the ring $\mathbb{F}_2[\mathbb{Z}^d]$.

In the next section, after a preliminary formulation of our main result (Theorem 1), we pass to a more detailed investigation of the duality map and to a discussion of a number of examples. In particular, we find that some self-dual systems are 'more self-dual' then others, leading to their classification into three types, called here A, B, C, of a descending degree of self-duality. While there is an infinite number of models of each type, and a few models of the first two types have been solved, the author is not aware of a soluble model of Type C. The distinction between different class of models acquires even more substance when one considers finite systems with periodic boundary conditions, associated error correcting codes, and zeros of partition function (see [19]). Section 3 contains proofs of the results formulated in Sect. 2.

A simple version of the main result of the present paper, Theorem 1 below, was described in [17]. This note did not make it through the refereeing process, but the result was mentioned in [18, concluding remarks] and [7, p. 25]. Since then a number of models covered by our results have been shown to be self-dual (see models M_2 of Sect. 2). Our interest in the subject has been rekindled by its relation to coding theory, both classical and quantum, which is discussed in [19].

2 The results

A d-dimensional spin $1/2$ ferromagnetic system with a finite range interaction is *transitive* if its energy can be written as

$$H = -\sum_{B \in \mathcal{B}} J(B)\sigma_B, \quad \text{where } \sigma_B = \prod_{a \in B} \sigma_a, \ \sigma_a = \pm 1, \ a \in \mathbb{Z}^d, \ J(B) > 0,$$

where $\mathcal{B}$ is a $\mathbb{Z}^d$-*invariant* family of finite subsets of the lattice $\mathbb{L} = \mathbb{Z}^d$ (*bonds*) with a *finite fundamental subfamily*[1]; we call such a system transitive since the translations under which the set of the bonds is invariant act on the lattice in a transitive way. In most of the paper, the dimension d of the lattice is larger than 1 and the *interaction* J is periodic, i.e., invariant under a subgroup of $\mathbb{Z}^d$ of a finite index—we refer to p. 9 below for a discussion of a number of examples.

Let

$$Z_\Lambda(K) = \sum_{\sigma_a = \pm 1, a \in \Lambda} \exp \sum_{B \in \mathcal{B}, B \subset \Lambda} K(B)\sigma_B, \quad K(B) := \frac{J(B)}{k_{\text{Boltzmann}} T} = \beta J(B), \tag{1}$$

be the partition function with 'zero boundary conditions' of a system in a finite subset Λ of the lattice, and let

$$P(K) = \lim_{\Lambda} \frac{1}{|\Lambda|} \ln Z_\Lambda(K)$$

be the (dimensionless) free energy ('pressure') of the system. We will refer also to K as the interaction of the system.

While the full description of our results (Theorem 2 of the end of this section) is somewhat involved, we start with the simplest form of the main result—the usual duality relation for the free energy for translation invariant interaction:

Theorem 1 *Suppose that all the bonds of a system are translations of just two*, B_1 *and* B_2, *and suppose that the interaction of the system is translation invariant. Define a translation invariant interaction* K^* *with the same bonds* $\mathcal{B}$ *by*

$$\tanh K^*(B_1) = \exp -2K(B_2), \quad \tanh K^*(B_2) = \exp -2K(B_1).$$

Then

$$P(K) - \frac{1}{2}(\ln\cosh 2K_1 + \ln\cosh 2K_2) = P(K^*) - \frac{1}{2}\left(\ln\cosh 2K_1^* + \ln\cosh 2K_2^*\right). \tag{2}$$

The duality relation (2) can also be cast into a rational form: Let $w = (w_1, w_2) = (e^{-2K_1}, e^{-2K_2})$ and let $w \mapsto \tilde{P}(w)$ be the function of w defined by the left hand side of (2). Then (2) is equivalent to

$$\tilde{P}\left(\left(\frac{1-w_2}{1+w_2}, \frac{1-w_1}{1+w_1}\right)\right) = \tilde{P}((w_1, w_2)).$$

We pass now to a more complete discussion of our result and of a number of examples.

Following [12, 22], for a finite subset Λ of the lattice, we set $\mathcal{B}_\Lambda = \{B \in \mathcal{B} : B \subset \Lambda\}$, $\Omega_\Lambda = \{-1, 1\}^\Lambda$, and we let γ_Λ, Γ_Λ and $\mathcal{K}_\Lambda$ be defined by the formulae (6)–(9) with $\mathcal{B}$ and Ω replaced by $\mathcal{B}_\Lambda$ and Ω_Λ, respectively. Furthermore, let $\mathcal{S}_\Lambda = \ker \gamma_\Lambda$ and let

$$Z_\Lambda = |\mathcal{S}_\Lambda| \left(\prod_{B \in \mathcal{B}_\Lambda} \exp K(B)\right) Z_\Lambda^{\text{LT}} \quad \text{where } Z_\Lambda^{\text{LT}} = \sum_{\beta \in \Gamma_\Lambda} \prod_{B \in \beta} e^{-2K(B)}, \tag{3}$$

[1] A subfamily $\mathcal{B}_0$ of $\mathcal{B}$ is *fundamental* if any element of $\mathcal{B}$ is a translation of just one of its elements.

$$Z_\Lambda = |\Omega_\Lambda| \left(\prod_{B \in \mathcal{B}_\Lambda} \cosh K(B) \right) Z_\Lambda^{\mathrm{HT}} \quad \text{where } Z_\Lambda^{\mathrm{HT}} = \sum_{\beta \in \mathcal{K}_\Lambda} \prod_{B \in \beta} \tanh K(B), \tag{4}$$

be the Low Temperature (LT) and High Temperature (HT) Expansions of the partition function (1); as in [7, 21, 22] and other works, self-duality is obtained once one finds one-to-one transformation $\mathcal{B} \to \mathcal{B}$ which maps $\mathcal{K}$ onto Γ. Moreover, the transformation should respect the translation invariance of the system. The essential difference in the approach of [7, 22] and that of the present work is that while in these references one uses the group, or of a vector space, structure of $\mathcal{K}$ and Γ, and the translation invariance is obtained as an afterthought, here, following [8], one takes advantage of the action of translations to enrich the algebraic structure, after which the self-duality is rather obvious.

We begin by recalling a few definitions. $\mathcal{P}(\mathcal{B})$ ($\mathcal{P}_f(\mathcal{B})$) denotes the family of all subsets (all finite subsets) of $\mathcal{B}$. $\mathcal{P}(\mathcal{B})$, and $\mathcal{P}_f(\mathcal{B})$, is an Abelian group under the operation $\alpha, \beta \mapsto \alpha + \beta$ of symmetric difference:

$$\alpha + \beta = (\alpha \backslash \beta) \cup (\beta \backslash \alpha), \tag{5}$$

the empty subset being the zero-element of the group. Since $2\alpha = 0$ for any α, $\mathcal{P}(\mathcal{B})$ is a vector space over $\mathbb{F}_2$.

As mentioned earlier, we assume that the family $\mathcal{B}$ of bonds is translation invariant, i.e., for any $B \in \mathcal{B}$, its translation by $a \in \mathbb{Z}^d$, $\tau_a B$, is also an element of $\mathcal{B}$. τ will also stand for action of $\mathbb{Z}^d$ on $\mathcal{P}_f(\mathcal{B})$ and other objects derived from $\mathcal{B}$. Subgroups of $\mathcal{P}_f(\mathcal{B})$ which are invariant under translations (by vectors of $\mathbb{Z}^d$) are called $\mathbb{Z}^d$-*modules* or $\mathbb{F}_2[\mathbb{Z}^d]$-*modules*.

Thus, following [8], we consider the $\mathbb{Z}^d$-modules of 'cycles'

$$\mathcal{K}_f = \mathcal{K}_f(\mathcal{B}) = \{\beta \in \mathcal{P}_f(\mathcal{B}) : \textstyle\prod_{B \in \beta} \sigma_B = 1 \text{ for any } \sigma \in \Omega\}, \tag{6}$$

and of 'contours'

$$\Gamma_f = \Gamma_f(\mathcal{B}) = \{\beta \in \mathcal{P}(\mathcal{B}) : \beta = \gamma(\sigma) \text{ for some } \sigma \in \Omega_f\}, \tag{7}$$

$$(\Gamma)_f = (\Gamma)_f(\mathcal{B}) = \{\beta \in \mathcal{P}(\mathcal{B}) : \beta = \gamma(\sigma) \text{ for some } \sigma \in \Omega\} \cap \mathcal{P}_f(\mathcal{B}); \tag{8}$$

($(\Gamma)_f$ is the $\mathrm{cl}\Gamma_f$ of [8]); here $\Omega = \{-1, 1\}^{\mathbb{L}}$ is the set of all the spin configurations, Ω_f is the subfamily of configurations with "+"-boundary conditions ($\Omega_f = \{\sigma \in \Omega :$ the set of $a \in \mathbb{L}$ for which $\sigma_a = -1$ is finite$\}$), and γ,

$$\gamma(\sigma) = \{B \in \mathcal{B} : \sigma_B = -1\}, \quad \sigma \in \Omega, \tag{9}$$

is a $\mathbb{Z}^d$-modules homomorphism $\Omega \to \mathcal{P}(\mathcal{B})$.

Proposition 1 *For any transitive system*, $\Gamma_f(\mathcal{B})$ (*and* $(\Gamma)_f$) *is always generated* (*as* $\mathbb{Z}^d$-*modules*) *by one of its elements*; $\mathcal{K}_f(\mathcal{B})$ *is generated by just one of its elements if and only if* $|\mathcal{B}_0| = 2$. *The generator of* $\Gamma_f(\mathcal{B})$ (*and of* $(\Gamma)_f$) *is unique*, *up to a translation*. *The same is true about the generator of* $\mathcal{K}_f$ *in case of* $|\mathcal{B}_0| = 2$. (*In other words*, *the* $\mathbb{Z}^d$-*module* $\Gamma_f(\mathcal{B})$ *is always* (*a free module*) *of rank one*, *and the module* $\mathcal{K}_f(\mathcal{B})$ *is of rank one if and only if the module* $\mathcal{B}$ *is of rank* 2.)

In the rest of this section it is assumed that all the bonds $\mathcal{B}$ of the system are translations of just two (*two-bond system*), and we fix arbitrarily a fundamental subfamily $\mathcal{B}_0 = \{B_1, B_2\}$ of $\mathcal{B}$.

We will say that an (ordered) pair of subsets of the lattice is a *translation* of another pair of subsets, say (A_1, A_2) is a translation of (A'_1, A'_2), if $A_i = \tau_a A'_i$, $i = 1, 2$, for some $a \in \mathbb{Z}^d$, and we write then $(A_1, A_2) = \tau_a(A'_1, A'_2)$; and for any subset A of $\mathbb{Z}^d$, its *inversion* $I(A)$ is defined by

$$I(A) = \{a \in \mathbb{Z}^d : -a \in A\}; \tag{10}$$

we set $I(\mathcal{B}) = \{I(B) : B \in \mathcal{B}\}$.

If $\mathcal{B}$ and $\mathcal{B}^*$ are two sets with $\mathbb{Z}^d$ action, then a one-to-one mapping $B \mapsto B^*$ of $\mathcal{B}$ onto $\mathcal{B}^*$ such that $(\tau_a B)^* = \tau_a(B^*)$ $((\tau_a B)^* = \tau_{-a}(B^*))$ for all $a \in \mathbb{Z}^d$ will be said to commute (anticommute) with translations and will be called $\mathbb{Z}^d$*-bijection* ($I\mathbb{Z}^d$*-bijection*). $\mathbb{Z}^d$-bijection ($I\mathbb{Z}^d$-bijection) $B \mapsto B^*$ induces $\mathbb{Z}^d$*-isomorphisms* ($I\mathbb{Z}^d$*-isomorphisms*) of subgroups of $\mathcal{P}_f(\mathcal{B})$ onto subgroups of $\mathcal{P}_f(\mathcal{B}^*)$ in the standard way. In particular, the mapping $B \mapsto I(B)$ induces $I\mathbb{Z}^d$-isomorphisms of subgroups of $\mathcal{P}_f(\mathcal{B})$ onto subgroups of $\mathcal{P}_f(I(B))$.

In general, $(\Gamma)_f$ may be strictly larger than Γ_f. A system is *reduced*, [8], if $(\Gamma)_f = \Gamma_f$. Every system is equivalent, in a sense, to a reduced one—its *reduced version*, [8]. The next four propositions are concerned with isomorphisms of the groups $\mathcal{K}_f$ and $(\Gamma)_f$ which commute, or anticommute, with translations. In Propositions 2–4 and the Corollary to Proposition 2, it is assumed that one is dealing with *a reduced transitive two-bond system*. Proposition 5 extends these results to general, not necessarily reduced, two-bond systems.

Proposition 2 *A $\mathbb{Z}^d$-bijection $B \mapsto B^*$ of $\mathcal{B}$ onto $I(\mathcal{B})$ induces a $\mathbb{Z}^d$-isomorphism of $\mathcal{K}_f(\mathcal{B})$ onto $\Gamma_f(I(\mathcal{B}))$ if and only if (B_1^*, B_2^*) is a translation of $(I(B_2), I(B_1))$.*

Corollary *There exists an $I\mathbb{Z}^d$-bijection of $\mathcal{B}$ onto $\mathcal{B}$ which induces an $I\mathbb{Z}^d$-isomorphism of $\mathcal{K}_f(\mathcal{B})$ onto $\Gamma_f(\mathcal{B})$. The $I\mathbb{Z}^d$-bijection is unique, up to a translation. If $\mathcal{B}_0 = \{B_1, B_2\}$ is a fundamental subfamily of $\mathcal{B}$ then a self-duality mapping $B \mapsto B^*$ can be obtained by setting $B_1^* = B_2$, $B_2^* = B_1$, and then by extending the mapping to all of $\mathcal{B}$ so that it anticommutes with translations.*

Taken together, Propositions 1–2 will imply that a transitive ferromagnetic lattice system is self-dual if and only if it is a two-bond system.

Proposition 3 *Let $B \mapsto B^*$ be a $\mathbb{Z}^d$-bijection of $\mathcal{B}$ onto $\mathcal{B}$ which induces an isomorphism of $\mathcal{K}_f(\mathcal{B})$ onto $\Gamma_f(\mathcal{B})$. Then $\mathcal{B}$ is invariant under I (i.e., $I(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$), (B_1^*, B_2^*) is a translation of $(I(B_2), I(B_1))$, and either*

(a) *B_1^* is a translation of B_2 and B_2^* is a translation of B_1, i.e., $I(B_i)$ is a translation of B_i, $i = 1, 2$. (Examples M_1, M_2 and M_4 below), or*
(b) *(B_1^*, B_2^*) is a translation of (B_1, B_2), and then B_2 is a translation of $I(B_1)$ (and B_1 is a translation of $I(B_2)$). (Example M_3 below.)*

We note that in both (a)- and (b)-cases, $\mathcal{B}$ is I-invariant and B_i^* is a translation of $I(B_j)$, $i \neq j$.

Definition A subset B of $\mathbb{Z}^d$ is *I-symmetric* if it is invariant under an inversion with respect to a point of $\frac{1}{2}\mathbb{Z}^d$, i.e., if $I(\tau_{-a}B) = \tau_{-a}B$ for some $a \in \frac{1}{2}\mathbb{Z}^d$. A two-bond model is of *Type A* if B_2 is a translation of $I(B_1)$ (Examples M_3 and M_6, below), of *Type B* if its bonds are I-symmetric (examples M_1, M_2, M_4, M_5, below), of *Type A∨B* if it is either of Type A or of Type B, and of *Type C* if it is not of Type A∨B.

Hence, a model is of Type A∨B if and only if its family of bonds is I-invariant.

The next Proposition shows that Type A systems are self-dual in a strongest possible sense and gives a geometric characterization of Type B systems.

Proposition 4

(i) *There exists a* $\mathbb{Z}^d$*-bijection of* $\mathcal{B}$ *onto* $\mathcal{B}$ *which induces an isomorphism of* $\mathcal{K}_f(\mathcal{B})$ *onto* $\Gamma_f(\mathcal{B})$ *if and only if the system is of Type* A ∨ *B*

(ii) *The following three conditions are equivalent*:

  (a) *The system is of Type A*

  (b) $\mathcal{K}_f = \Gamma_f$

  (c) $\mathcal{K}_f$ *and* Γ_f *have non-trivial intersection, i.e.*, $\mathcal{K}_f \cap \Gamma_f \neq \{\mathbf{0}\}$

(iii) *A system is of Type B if and only if its bonds are* I*-symmetric sets. And no system is both of Type A and Type B.*

Proposition 5 (*With an obvious modification of definitions*) *Propositions* 2–4 *and Corollary to Propositions* 2 *hold for all, not necessarily reduced, transitive two-bond systems, provided* Γ_f *is replaced by* $(\Gamma)_f$.

Definition A $\mathbb{Z}^d$-system $(\mathbb{L}, \mathcal{B}, K)$ is **self-dual** if there is a $\mathbb{Z}^d$- or an $I\mathbb{Z}^d$-bijection $B \mapsto B^*$ from $\mathcal{B}$ onto $\mathcal{B}$ ('self-duality mapping'), such that the induced isomorphism of $\mathcal{P}_f(\mathcal{B})$ onto $\mathcal{P}_f(\mathcal{B})$ maps $\mathcal{K}_f(\mathcal{B})$ onto $\Gamma_f(\mathcal{B})$, or onto $(\Gamma)_f(\mathcal{B})$, if the system is not reduced.

Defining, for a finite $\Lambda \subset \mathbb{L}$, the 'symmetrized' partition function and pressure:

$$\tilde{Z}_\Lambda(K) = \frac{Z_\Lambda(K)}{\sqrt{|\Omega_\Lambda|\,|\mathcal{S}_\Lambda|\prod_{B\in\mathcal{B},B\subset\Lambda}\cosh 2K(B)}} \quad \text{and}$$

$$\tilde{P}(K) = \lim_\Lambda |\Lambda|^{-1}\ln\tilde{Z}_\Lambda(K)$$

—$\tilde{Z}$ is the Y of [8, Appendix A] and [22, (2.24)], for systems with a periodic K, say $n\mathbb{Z}^d$-invariant, one has

$$\tilde{P}(K) = P(K) - \frac{1}{2n^d}\sum_{B\in\mathcal{B}/n\mathbb{Z}^d}\ln\cosh 2K(B), \tag{11}$$

and if such a system is self-dual then

$$\tilde{P}(K^*) = \tilde{P}(K), \tag{12}$$

where K^* is related to K by $\tanh K^*(B^*) = \exp -2K(B)$, [7, 22]. For translation invariant K, (12) reduces to (2); (12) holds also for quasiperiodic K.

The main results of Propositions 1–5 can now be summarized as follows:

Theorem 2 *A transitive ferromagnetic spin* 1/2 *system is self-dual if and only if it is a two-bond system. For any two-bond system, there exists a unique, up to translation, self-duality* $I\mathbb{Z}^d$*-mapping, and there exists a self-duality* $\mathbb{Z}^d$*-mapping if and only if the system is of Type* $A \vee B$.

Examples We place now some well known models in the framework of the present paper and show that self-dual models can be quite varied. More detailed analysis is obtained when one considers properties of finite volume systems with periodic boundary conditions, [19].

The last two models, M_5 and M_6, are reduced three-dimensional. As is typical for models in dimension higher than two, they have infinite number of extremal Gibbs states at low temperatures. Discussion of more models, especially of Type A, will be found in [19].

We will write $\mathbf{x}$, $\mathbf{y}$ ($\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$) for the elements of the canonical basis of $\mathbb{Z}^2$ ($\mathbb{Z}^3$).

(M_1) **Two-dimensional Ising Model**:

$$B_1 = \{\mathbf{0}, \mathbf{x}\} = \bullet \quad \bullet, \qquad B_2 = \{\mathbf{0}, \mathbf{y}\} = \begin{matrix}\bullet\\ \bullet\end{matrix}.$$

The model is reduced and of Type B. The $\mathbb{Z}^2$-modules $\mathcal{K}_f$ and Γ_f are generated by $\beta_0 = \{B_1, \tau_{\mathbf{y}} B_1, B_2, \tau_{\mathbf{x}} B_2\}$ (‘square’) and $\gamma_0 = \{B_1, \tau_{-\mathbf{x}} B_1, B_2, \tau_{-\mathbf{y}} B_2\}$ (‘cross’) respectively. In the notation of the formulae (15) and (17), below,

$$\beta_0 = B_2 \cdot \{B_1\} + B_1 \cdot \{B_2\} \quad \text{and} \quad \gamma_0 = I(B_1) \cdot \{B_1\} + I(B_2) \cdot \{B_2\}.$$

(M_1') This is a **blown-up version of the two-dimensional Ising Model** (i.e., M_1 is its reduced version):

$$B_1' = \{\mathbf{0}, 2\mathbf{x}\} = \bullet \quad \circ \quad \bullet, \qquad B_2' = \{\mathbf{0}, \mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y}\} = \begin{matrix}\bullet & \bullet\\ \bullet & \bullet\end{matrix}$$

(the circle indicates a point that is not in the bond). The $\mathbb{Z}^2$-modules $\mathcal{K}_f$ and $(\Gamma)_f$ are generated by $\beta_0 = \{B_1', \tau_{\mathbf{y}} B_1', B_2', \tau_{\mathbf{x}} B_2'\}$ and $\gamma_0 = \{B_1', \tau_{-\mathbf{x}} B_1', B_2', \tau_{-\mathbf{y}} B_2'\}$, respectively, while $\gamma_0' = \{B_1', \tau_{-2\mathbf{x}} B_1', B_2', \tau_{-\mathbf{x}} B_2', \tau_{-\mathbf{y}} B_2', \tau_{-(\mathbf{x}+\mathbf{y})} B_2'\}$ is a generator of Γ_f. In the notation of the formulae (15), (16) and (17), below, $D = \gcd(B_1', B_2') = \{0, \mathbf{x}\}$, $B_i = B_i'/D$, $\beta_0 = B_2 \cdot \{B_1'\} + B_1 \cdot \{B_2'\}$ and $\gamma_0 = I(B_1) \cdot \{B_1'\} + I(B_2) \cdot \{B_2'\}$, respectively, while $\gamma_0' = I(B_1') \cdot \{B_1'\} + I(B_2') \cdot \{B_2'\}$, and B_1, B_2 are as in M_1. The model has an infinite number of ground states—one can flip spins, independently in each row, but still only two pure phases at low enough temperatures, [8].

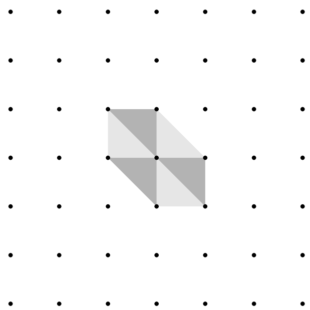
A detailed analysis of a three-dimensional blowned-up version of the two-dimensional Ising Model can be found in [16], where it was noted, among other things, that the model is self-dual.

(M_2) Two-dimensional Type B **model with a two- and three-spin interaction**: $B_1 = \{0, \mathbf{x}, 2\mathbf{x}\}$ and $B_2 = \{\mathbf{0}, \mathbf{y}\}$ ([1, 2, 6, 15]); [20] has its $m \times n$ generalization with $B_1 = \{\mathbf{0}, \mathbf{x}, 2\mathbf{x}, \ldots, m\mathbf{x}\}$ and $B_2 = \{0, \mathbf{y}, 2\mathbf{y}, \ldots, n\mathbf{y}\}$. A further generalization is obtained by considering ‘linear bonds’ with gaps. If the bonds are inversion invariant, like in the case of $B_1 = \bullet\ \bullet\ \circ\ \bullet\ \bullet$ and $B_2 = \begin{matrix}\bullet\\ \bullet\end{matrix}$, one obtains a Type B model, while its slight variation

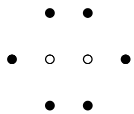
$$B_1 = \bullet \quad \bullet \quad \bullet \quad \circ \quad \bullet \quad \text{and} \quad B_2 = \begin{matrix}\bullet\\ \bullet\end{matrix}$$

yields a model, M_3', which is of Type C.

(M_3) **Triangular Three Spin**, or Baxter-Wu, Model: $B_1 = \{\mathbf{0}, \mathbf{x}, \mathbf{y}\}$, $B_2 = \{\mathbf{0}, -\mathbf{x}, -\mathbf{y}\} = I(B_1)$—the simplest model of Type A.



The model has been introduced and shown to be self-dual in [12] and [24], and then solved in [3, 4].
(M_4) Here is a more complicated Type B model: $B_1 = \{\mathbf{0}, \mathbf{x}, 2\mathbf{x}+\mathbf{y}, \mathbf{x}+2\mathbf{y}, 2\mathbf{y}, -\mathbf{x}+\mathbf{y}\}$, $B_2 = \{\mathbf{0}, \mathbf{y}\}$,

$B_1 =$ , $B_2 =$ .

The model is reduced; its ground states are the same as that of 'linear' system with $\mathcal{B}_0 = \{\{0, 3\mathbf{x}\}, \{0, \mathbf{y}\}\}$, which implies that it has 2^3 ground states, and the same number 2^3 of pure phases at low temperatures.
(M_5) A three-dimensional reduced Type B model: $B_1 = \{\mathbf{0}, \mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y}\}$, $B_2 = \{\mathbf{0}, \mathbf{z}\}$.
(M_6) A three-dimensional reduced Type A ('simplex') model: $B_1 = \{\mathbf{0}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$, $B_2 = I(B_1)$.

3 Proofs

In the proofs, we will use the algebraic structure introduced into the present context in [8]. We refer to [8] for a more detailed exposition.

The Abelian group $\mathcal{P}_f(\mathbb{Z}^d)$ (under symmetric difference (5)) when equipped with the multiplication operation $A \cdot B = \sum_{a \in A} \tau_a B$ is a ring, denoted by $\mathcal{R}$ in what follows—the group algebra $\mathbb{F}_2[\mathbb{Z}^d]$ of $\mathbb{Z}^d$ with coefficients in the two element field $\mathbb{F}_2 = \{0, 1\}$. Any $\mathbb{Z}^d$-module is a $\mathcal{R}$ *-module*. The modules considered below are $\mathcal{P}_f(\mathcal{B})$ and its submodules $\mathcal{K}_f$, Γ_f and $(\Gamma)_f$. The product of an element β of $\mathcal{P}_f(\mathcal{B})$ by an element A of $\mathcal{R}$ is

$$A \cdot \beta = \sum_{a \in A} \tau_a \beta. \tag{13}$$

If $\mathcal{B}$ and $\mathcal{B}^*$ are two sets with $\mathbb{Z}^d$ action and $B \mapsto B^*$ is a $\mathbb{Z}^d$-mapping $\mathcal{B} \to \mathcal{B}^*$ then the induced mapping $\mathcal{P}_f(\mathcal{B}) \to \mathcal{P}_f(\mathcal{B}^*)$ commutes with the action of the ring:

$$(A \cdot \beta)^* = A \cdot \beta^*. \tag{14}$$

We will use the following, 'polynomial', notation ([8]): for $a \in \mathbb{Z}^d$, we will write X^a for the element $\{a\}$ of $\mathcal{R}$ and then $X^a \beta$ for the element $\{a\} \cdot \beta$, where β is an element of an $\mathcal{R}$-module. We have then $X^{a+b} = X^a X^b$.

We will use the following simple properties of the ring $\mathcal{R}$, [8]:

(P1) It has no *zero divisors*: if $P \cdot Q = 0$, $P, Q \in \mathcal{R}$, then either $P = 0$ or $Q = 0$.
(P2) *units* (invertible elements) of $\mathcal{R}$ are of the form $\{a\}$, $a \in \mathbb{Z}^d$.
(P3) (Existence of a *greatest common divisor*) for any family $(P_i : i \in I)$ of elements of $\mathcal{R}$ there exists a unique, up to a factor which is a unit, greatest common divisor of the family, $\gcd(P_i : i \in I)$, i.e., an element P of $\mathcal{R}$ which divides each P_i and such that any other element of $\mathcal{R}$ with this property divides P.

For a two-bond system, the unique, up to translation, generator of the $\mathbb{Z}^d$-module $\mathcal{K}_f(\mathcal{B})$ will be denoted β_0. In case of a reduced system

$$\beta_0 = \sum_{a \in B_1} \{\tau_a B_2\} + \sum_{a \in B_2} \{\tau_a B_1\},$$

or, in the notation of (14),

$$\beta_0 = B_1 \cdot \{B_2\} + B_2 \cdot \{B_1\}; \tag{15}$$

in general, when the system is not necessarily reduced,

$$\beta_0 = \frac{B_1}{D} \cdot \{B_2\} + \frac{B_2}{D} \cdot \{B_1\} \quad \text{where } D = \gcd(B_1, B_2). \tag{16}$$

$\Gamma_f(\mathcal{B})$ is always generated by

$$\gamma_0 = I(B_1) \cdot \{B_1\} + I(B_2) \cdot \{B_2\}, \tag{17}$$

while $(\Gamma)_f$ is generated by

$$\gamma_0 = I\left(\frac{B_1}{D}\right) \cdot \{B_1\} + I\left(\frac{B_2}{D}\right) \cdot \{B_2\}. \tag{18}$$

Proof of Proposition 1 If $|\mathcal{B}_0| > 2$ then, according to app. B of [8], $\mathcal{K}_f$ is not generated by a one element (i.e., the $\mathcal{R}$-module $\mathcal{K}_f$ is of rank larger than 1), while Γ_f is, and therefore the system is not self-dual. (An earlier, somewhat different formulation of the proof of the fact that no system with $|\mathcal{B}_0| > 2$ is self-dual, can be found in [7, p. 25].)

The uniqueness part follows from the following remark: let

$$\alpha = \sum_{i=1}^{k} A_i \{B_i\}$$

be an elements of $\mathcal{P}_f(\mathcal{B})$, and let $\alpha' = C \sum_{i=1}^{k} A_i \{B_i\}$ be a generator of the submodule $\mathcal{R} \cdot \alpha$ of $\mathcal{P}_f(\mathcal{B})$ generated by α. Then there exists an $A \in \mathcal{R}$ such that $\alpha = A\alpha'$, i.e., such that $AC = 1$. It follows that C is a unit. ("Generator of a singly generated submodule of a free module is unique, up to a factor which is a unit.") To finish the proof, it is enough to apply this remark to Γ_f, $(\Gamma)_f$ (for any transitive system) and to $\mathcal{K}_f$ (for any transitive two-bond system). $\square$

Proof of Proposition 2 Let $B \mapsto B^*$ be as in the Proposition. Both B_1^* and B_2^* cannot be translations of the same element of $I(\mathcal{B}_0)$ since then B_2^* would be a translation of B_1^*, and therefore B_2 would be a translation of B_1. Hence either (a) B_1^* is a translation of $I(B_2)$ and B_2^* is a translation of $I(B_1)$, or (b) B_1^* is a translation of $I(B_1)$ and B_2^* is a translation of $I(B_2)$.

If (b) is the case, say $B_i^* = X^{a_i} I(B_i)$ for some $a_1, a_2 \in \mathbb{Z}^d$, then

$$\begin{aligned}\beta_0^* &= (B_2\{B_1\} + B_1\{B_2\})^* \\ &\overset{\text{by (14)}}{=} B_2\{B_1^*\} + B_1\{B_2^*\} = B_2\{X^{a_1} I(B_1)\} + B_1\{X^{a_2} I(B_2)\}\end{aligned}$$

is a generator of Γ_f since β_0 is a generator of $\mathcal{K}_f$ ($|\mathcal{B}_0| = 2!$). It then follows from the uniqueness part of Proposition 1 that there exists $a \in \mathbb{Z}^d$ such that

$$B_2\{X^{a_1} I(B_1)\} + B_1\{X^{a_2} I(B_2)\} = X^a \left(B_1\{I(B_1)\} + B_2\{I(B_2)\}\right), \tag{19}$$

which implies that B_2 is a translation of B_1—a contradiction proving that only the case (a) can occur.

Let then $B_1^* = X^{a_1} I(B_2)$ and $B_2^* = X^{a_2} I(B_1)$ for some $a_1, a_2 \in \mathbb{Z}^d$. Then (19) is replaced by

$$B_2\{X^{a_1} I(B_2)\} + B_1\{X^{a_2} I(B_1)\} = X^a \left(B_1\{I(B_1)\} + B_2\{I(B_2)\}\right),$$

which implies that $a_2 = a_1$, as claimed.

To prove the "if-part", let $B \mapsto B^*$ be the (unique) $\mathbb{Z}^d$-bijection of $\mathcal{B}$ onto $\mathcal{B}^* := I(\mathcal{B})$ such that $B_i^* = I(B_j)$, $i \neq j$. Denoting again by $*$ the induced $\mathbb{Z}^d$-isomorphism of $\mathcal{P}_f(\mathcal{B})$ onto $\mathcal{P}_f(\mathcal{B}^*)$, one obtains

$$\beta_0^* = (B_2 \cdot \{B_1\} + B_1 \cdot \{B_2\})^* = B_2 \cdot \{B_1^*\} + B_1 \cdot \{B_2^*\} = I(B_1^*) \cdot \{B_1^*\} + I(B_2^*) \cdot \{B_2^*\}.$$

Since $I(B_1^*) \cdot \{B_1^*\} + I(B_2^*) \cdot \{B_2^*\}$ is a generator of $\Gamma_f(\mathcal{B}^*)$, $B \mapsto B^*$ defines an $\mathbb{Z}^d$-isomorphism of $\mathcal{K}_f(\mathcal{B})$ onto $\Gamma_f(\mathcal{B}^*)$. This ends a proof of Proposition 3. □

Proof of Corollary $\varphi : \beta \mapsto I(\beta) := \{I(B) : B \in \mathcal{B}\}$ is an $I\mathbb{Z}^d$-isomorphism of $\mathcal{P}_f(\mathcal{B})$ onto $\mathcal{P}_f(I(\mathcal{B}))$. It maps $A \cdot \beta$ onto $I(A) \cdot I(\beta)$, and therefore $\varphi(\gamma_0)$ is a generator of $\Gamma_f(I(\mathcal{B}))$. □

Proof of Proposition 3 Arguing as in the proof of Proposition 2, we see that either (a) B_1^* is a translation of B_2 and B_2^* is a translation of B_1, or (b) B_1^* is a translation of B_1 and B_2^* is a translation of B_2.

Suppose that one has the case (b), say $B_i^* = X^{a_i} B_i$ for some $a_i \in \mathbb{Z}^d$, $i = 1, 2$. Then instead of (19) one has

$$B_2\{X^{a_1} B_1\} + B_1\{X^{a_2} B_2\} = X^a \left(I(B_1)\{B_1\} + I(B_2)\{B_2\}\right) \tag{20}$$

for some $a \in \mathbb{Z}^d$. It follows that

$$X^{a_1} B_2 = X^a I(B_1) \quad \text{and} \quad X^{a_2} B_1 = X^a I(B_2). \tag{21}$$

(21), which is the same as $(B_1^*, B_2^*) = \tau_a(I(B_2), I(B_1))$ and which implies that $a_2 = a_1$, and then that

$$B_2 = X^c I(B_1) \quad (\text{and } B_1 = X^c I(B_2)), \quad \text{where } c = a - a_1 = a - a_2,$$

proving (b). (We note that one can always choose $\mathcal{B}_0$ so that $B_2 = I(B_1)$, as we did in Example M_3.)

In the case of (a), say $B_1^* = X^{a_1} B_2$, $B_2^* = X^{a_2} B_1$, $a_1, a_2 \in \mathbb{Z}^d$, proceeding as in the (b)-case, one arrives at

$$B_2\{X^{a_1} B_2\} + B_1\{X^{a_2} B_1\} = X^a \left(I(B_1)\{B_1\} + I(B_2)\{B_2\} \right),$$

$a \in \mathbb{Z}^d$, i.e., at

$$X^{a_2} B_1 = X^a I(B_1) \quad \text{and} \quad X^{a_1} B_2 = X^a I(B_2),$$

which again can be written as $(B_1^*, B_2^*) = \tau_a(I(B_2), I(B_1))$, and which is the same as

$$I(B_1) = X^{a_2 - a} B_1 \quad \text{and} \quad I(B_2) = X^{a_1 - a} B_2.$$

Note that, unlike in case (b), in the (a)-case it is in general not true that $a_1 = a_2$ (see Example M_1'). □

Remark The example of $B_1 = \{\mathbf{0}, \mathbf{x}, \mathbf{2x}\}$, $B_2 = \{\mathbf{0}, \mathbf{y}\}$ shows that, in general, one may not be able to choose a translation B_1' of B_1 and a translation B_2' of B_2 so that $(I(B_1'), I(B_2'))$ is a translation of (B_1', B_2').

Proof of Proposition 4 To prove the first part of the Proposition, it is enough to show that (c) implies that B_2 is a translation of $I(B_1)$.

Suppose that β is a non-zero element of both Γ_f and $\mathcal{K}_f$:

$$\begin{aligned} &\beta = A_1 \left(I(B_1) \cdot \{B_1\} + I(B_2) \cdot \{B_2\} \right) \quad \text{and} \\ &\beta = A_2 \left(B_2 \cdot \{B_1\} + B_1 \cdot \{B_2\} \right), \end{aligned} \tag{22}$$

where $A_1, A_2 \in \mathcal{R}$. (22) is equivalent to

$$A_1 I(B_1) = A_2 B_2 \quad \text{and} \quad A_1 I(B_2) = A_2 B_1, \tag{23}$$

which implies that

$$B_1 I(B_1) = B_2 I(B_2). \tag{24}$$

Setting $C = \gcd(I(B_1), B_2)$,

$$B_2' = B_2 / C \quad \text{and} \quad B_1' = B_1 / I(C), \tag{25}$$

(24) implies that $B_1' I(B_1') = B_2' I(B_2')$, and since $I(B_1')$ and B_2' are relatively prime, this implies that $B_2' = U \cdot B_1'$ for some $U \in \mathcal{R}$ for which $U \cdot I(U) = 1$, which in turn implies that U is a unit of $\mathcal{R}$, and that therefore B_2' is a translation of B_1'. Furthermore, by (25), $B_2 = U \cdot C \cdot B_1'$ and $B_1 = I(C) \cdot B_1'$. But since $\gcd(B_1, B_2)$ is a unit (the system is reduced!), this implies B_1' is a unit and that therefore B_2 is a translation of $I(B_1)$.

The second part of the Proposition is obvious since $I(B) = \tau_b B$ is equivalent to $\tau_a(I(\tau_{-a} B)) = B$, where $a = -\frac{1}{2} b$, and $B_2 = X^a I(B_1) = X^b I(B_2)$ if and only if $B_2 = X^c B_1$, $c = b - a$. □

Proof of Theorem 2 The "only if" part of the first statement of the theorem follows from Proposition 1, and of the second statement from Propositions 2 and 3. The uniqueness statement follows also from the propositions. To end the proof it remains to construct the duality mappings.

In case of Type A∨B system, define a self-duality map $\mathcal{B} \to \mathcal{B}$, $B \mapsto B^*$, as in the proof of Proposition 2 and then define K^* by

$$\tanh K^*(B^*) = \exp -2K(B), \quad B \in \mathcal{B}. \tag{26}$$

(Setting $w = e^{-2K(B)}$, $w^* = e^{-2K^*(B^*)}$, one obtains a homographic version of (26): $w^* = (1-w)/(1+w)$.)

In case of Type C systems, to obtain a self-duality map $B \mapsto B^*$, define first a $\mathbb{Z}^d$-bijection $B \mapsto B^{\sim}$ of $\mathcal{B}$ onto $I(\mathcal{B})$—the $B \mapsto B^*$ mapping of the proof Proposition 3 and then consider the $I\mathbb{Z}^d$-bijection $\mathcal{B} \ni B \mapsto B^* := I(B^{\sim}) \in \mathcal{B}$ with K^* defined by (26). This ends a proof of the Theorem. □

Proposition 5 is obtained through the reduction process of [8]: see formulae (16) and (18).

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